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# Transverse Diffusion of a Collimated Particle Beam

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We consider the problem of describing the steady-state spreading of a collimated particle beam as it penetrates a background material. The exact description for this problem is taken as the linear transport equation with full six-dimensional phase space dependence. In the limit of very forward peaked scattering with small energy transfer, the Fokker–Planck scattering description is used. To obtain a simplified model of beam transport, we assume that the beam in question has weak spatial gradients in the plane perpendicular to the beam direction, and that the beam nearly maintains its collimated integrity as it passes through the material. These assumptions lead to a hierarchy of advection-diffusion-like approximations for the spatial distribution of the particle density per unit energy. In the simple case of monoenergetic transport in a purely scattering homogeneous material, these equations are easily solved via Laplace and Fourier transformations to obtain explicit analytical results. Comparisons with benchmark Monte Carlo calculations give an indication of the accuracy of this treatment of beam spreading.

**KEY WORDS:** Advection-diffusion; beam spreading; beam transport; drift-diffusion; kinetic theory; transport theory; transverse diffusion.

## 1. INTRODUCTION

We consider the steady-state linear transport problem described by the equation

$$\begin{aligned}\boldsymbol{\Omega}\cdot\nabla\psi(\mathbf{r},E,\boldsymbol{\Omega})+\sigma(\mathbf{r},E)\,\psi(\mathbf{r},E,\boldsymbol{\Omega})&\\ =\int_0^\infty dE'\int_{4\pi}d\boldsymbol{\Omega}'\,\sigma_s(\mathbf{r},E'\to E,\boldsymbol{\Omega}'\cdot\boldsymbol{\Omega})\,\psi(\mathbf{r},E',\boldsymbol{\Omega}')&\\ -\infty<x,y<\infty,\qquad 0\leqslant z\leqslant T&\end{aligned}\tag{1}$$

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with boundary conditions

$$\psi(x, y, 0, E, \mu, \phi) = F(x, y, E)\frac{\delta(1-\mu)}{2\pi}, \qquad \mu > 0 \tag{2}$$

$$\psi(x, y, T, E, \mu, \phi) = 0, \qquad \mu < 0 \tag{3}$$

$$\psi(\pm\infty, y, z, E, \mu, \phi) = \psi(x, \pm\infty, z, E, \mu, \phi) = 0 \tag{4}$$

Here $F(x, y, E)$ is the specified boundary data, assumed to vanish as $x$ or $y$ increases without bound. The spatial variable $\mathbf{r}$ has Cartesian coordinates $x$, $y$, $z$, and the unit vector $\mathbf{\Omega}$, which points in the particle streaming direction, is described by a polar angle $\theta = \cos^{-1}(\mu)$ measured with respect to the $z$ axis, and by an azimuthal angle $\phi$ measured with respect to the $x$ axis. The symbol $E$ denotes the particle energy, and together $x$, $y$, $z$, $E$, $\mu$ and $\phi$ constitute the six coordinates of our phase space. The dependent variable $\psi(\mathbf{r}, E, \mathbf{\Omega}) = \psi(x, y, z, E, \mu, \phi)$ is the particle intensity, defined as the product of the particle speed and the six-dimensional phase space particle density. The total cross section and scattering kernel are denoted in Eq. (1) by $\sigma(\mathbf{r}, E)$ and $\sigma_s(\mathbf{r}, E' \to E, \mathbf{\Omega}' \cdot \mathbf{\Omega})$, respectively. As indicated by the angular arguments of these two quantities, we are assuming that the beam is transporting in an isotropic medium, where the probability of a particle collision is independent of the particle direction $\mathbf{\Omega}$, and the angular details of the scattering interaction depend only upon the scattering angle, $\cos^{-1}(\mathbf{\Omega}' \cdot \mathbf{\Omega})$.

The physics described by Eqs. (1)–(4) is that of steady-state transport of an extended (in $x$ and $y$) particle beam incident upon one side ($z = 0$) of an infinite (in $x$ and $y$) slab of thickness $T$. The spatial and energy dependences of this incident intensity are given by the function $F(x, y, E)$. No particles impinge upon the slab surface at $z = T$, and no particles are introduced in the interior of the slab. In writing Eq. (1), we have included a spatial dependence for both $\sigma$ and $\sigma_s$ to indicate that our treatment will allow an arbitrary spatial heterogeneity for the material comprising the slab. The beam incident at $z = 0$ is assumed to be fully collimated, as indicated by the Dirac delta function with argument $(1-\mu)$ in Eq. (2). Crucial to our analysis is the assumption that the $x$ and $y$ dependences of $F(x, y, E)$ are weak, which implies weak spatial transverse (in $x$ and $y$) gradients in the solution $\psi(\mathbf{r}, E, \mathbf{\Omega})$. Further, the slab is assumed thin, and/or the scattering is assumed to be very peaked in the forward direction, so that the beam exiting the slab at $z = T$ is only weakly perturbed from that entering at $z = 0$. In our analysis, we will explicitly neglect any backscattering of particles, and this also implies highly peaked forward scattering.

If one expands the scattering kernel in Legendre polynomials according to

$$\sigma_s(\mathbf{r}, E' \to E, \boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) = \sum_{n=0}^{\infty} \left(\frac{2n+1}{4\pi}\right) \sigma_{sn}(\mathbf{r}, E' \to E)\, P_n(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega}) \tag{5}$$

and uses the well-known addition formula for $P_n(\boldsymbol{\Omega}' \cdot \boldsymbol{\Omega})$, Eq. (1) can equivalently be written as[1,2]

$$\begin{aligned}
&\boldsymbol{\Omega} \cdot \nabla \psi(\mathbf{r}, E, \boldsymbol{\Omega}) + \sigma(\mathbf{r}, E)\, \psi(\mathbf{r}, E, \boldsymbol{\Omega}) \\
&\quad = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \int_0^{\infty} dE'\, \sigma_{sn}(\mathbf{r}, E' \to E) \int_0^{2\pi} d\phi' \int_{-1}^{1} d\mu'\, \psi(\mathbf{r}, E', \mu', \phi') \\
&\qquad \times \left[ P_n(\mu)\, P_n(\mu') + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\mu)\, P_n^m(\mu') \cos m(\phi - \phi') \right], \\
&\qquad 0 \leqslant z \leqslant T
\end{aligned} \tag{6}$$

where $P_n^m(\mu)$ is the associated Legendre function defined by

$$P_n^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m P_n(\mu)}{d\mu^m} \tag{7}$$

and the expansion coefficient $\sigma_{sn}$ in Eq. (5) is given by

$$\sigma_{sn}(\mathbf{r}, E' \to E) = 2\pi \int_{-1}^{1} d\xi\, P_n(\xi)\, \sigma_s(\mathbf{r}, E' \to E, \xi) \tag{8}$$

The physical content of Eq. (6) is identical to that of Eq. (1), with Eq. (6) simply being in a more explicit algebraic form.

If the scattering is sufficiently peaked in both angle (in the forward direction) and energy (about a zero energy exchange), the integral scattering operator in Eq. (6) can be asymptotically replaced by the Fokker–Planck differential operator.[3,4] The resulting transport equation is given by[1,2,5]

$$\begin{aligned}
&\boldsymbol{\Omega} \cdot \nabla \psi(\mathbf{r}, E, \boldsymbol{\Omega}) + \sigma_a(\mathbf{r}, E)\, \psi(\mathbf{r}, E, \boldsymbol{\Omega}) \\
&\quad = \frac{\sigma_1(\mathbf{r}, E)}{2} \left[ \frac{\partial}{\partial \mu} (1-\mu^2) \frac{\partial}{\partial \mu} + \left(\frac{1}{1-\mu^2}\right) \frac{\partial^2}{\partial \phi^2} \right] \psi(\mathbf{r}, E, \boldsymbol{\Omega}) \\
&\qquad + \frac{\partial}{\partial E} [S(\mathbf{r}, E)\, \psi(\mathbf{r}, E, \boldsymbol{\Omega})], \qquad 0 \leqslant z \leqslant T
\end{aligned} \tag{9}$$

Here $\sigma_a(\mathbf{r}, E)$ is the absorption cross section given by

$$\sigma_a(\mathbf{r}, E) = \sigma(\mathbf{r}, E) - 2\pi \int_{-1}^{1} d\xi \int_0^{\infty} dE'\, \sigma_s(\mathbf{r}, E \to E', \xi) \tag{10}$$

the cross section $\sigma_1(\mathbf{r}, E)$ is defined as

$$\sigma_1(\mathbf{r}, E) = 2\pi \int_{-1}^{1} d\xi (1-\xi) \int_0^\infty dE'\, \sigma_s(\mathbf{r}, E \to E', \xi) \tag{11}$$

and $S(E)$, the so-called stopping power, is given by

$$S(\mathbf{r}, E) = 2\pi \int_{-1}^{1} d\xi \int_0^\infty dE'(E-E')\, \sigma_s(\mathbf{r}, E \to E', \xi) \tag{12}$$

The sum of $\sigma_a$ and $\sigma_1$ is just the usual transport cross section $\sigma_{tr}$. In the radiological physics community, $2\sigma_1$ is denoted by $T$ and called the (linear) scattering power.[6] The symbol $T$ has also been used in the neutron transport theory literature,[2] but in this case $T = \sigma_1/2$. The Fokker–Planck description, given by Eq. (9) and applicable to problems involving very forward peaked scattering with small energy transfer,[5] is obviously algebraically much simpler than the full transport description given by Eq. (6). Applications involving the problem just described, either with the full integral scattering operator or with the differential Fokker–Planck operator, include cosmic ray transport, material modification by ion beams, and electron beams used in radiation therapy for cancer. References to these applications can be found in a recent paper by Börgers and Larsen.[7]

Our goal in the present paper is to develop a hierarchy of approximate descriptions for $\rho(\mathbf{r}, E)$ which, aside from a factor of particle speed, is just the particle density per unit energy. This quantity is defined as the angular integral of $\psi(\mathbf{r}, E, \mathbf{\Omega})$ according to

$$\rho(\mathbf{r}, E) = \int_{4\pi} d\mathbf{\Omega}\, \psi(\mathbf{r}, E, \mathbf{\Omega}) = \int_0^{2\pi} d\phi \int_{-1}^{1} d\mu\, \psi(\mathbf{r}, E, \mu, \phi) \tag{13}$$

That is, we want to eliminate the phase space variables $\mu$ and $\phi$ from the problem and obtain an accurate and simpler description for the integral quantity $\rho(\mathbf{r}, E)$. It is only $\rho(\mathbf{r}, E)$ which is needed to compute spatial profiles of collision rates and energy deposition; the angular detail found in $\psi(\mathbf{r}, E, \mathbf{\Omega})$ is not required for this purpose.

The earliest work along these lines seems to be due to Fermi,[8,9] who, in a seminar in 1940, obtained a very simple result in the case of purely scattering ($\sigma_a = 0$) monoenergetic transport in a homogeneous ($\sigma_s$ independent of space) medium. Such one-speed transport is characterized by the occurrence of a Dirac delta function, with argument $(E'-E)$, in the scattering kernel $\sigma_s(E' \to E, \mathbf{\Omega}' \cdot \mathbf{\Omega})$, Physically, this implies no particle energy change upon scattering, and thus particles at any given energy are

transported through the medium independently of those at other energies. Fermi's result is given by the equation

$$\frac{\partial \rho}{\partial z} = \frac{\sigma_1 z^2}{2} \nabla^2 \rho \tag{14}$$

where $\nabla^2$ is the transverse Laplacian given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{15}$$

The boundary conditions on Eq. (14) follow from Eqs. (2), (4), and (13) as

$$\rho(x, y, 0) = F(x, y) \tag{16}$$

together with the vanishing of $\rho(x, y, z)$ for large $x$ and $y$. In writing Eqs. (14) and (16) we have dropped the energy argument $E$ since it is simply a parameter in Fermi's result. We have also dropped the argument $\mathbf{r}$ to simplify the notation, a practice we will adopt henceforth. A written account of Fermi's solution can be found in Rossi and Greisen.[10] This work was later generalized by Eyges[11] to include small energy transfer in the scattering interaction. Very recently Larsen,[12] using an algebraically intensive procedure, showed that the Fermi result can be derived as an asymptotic limit of the Fokker–Planck description for this problem [see Eq. (9)]. In this work, the stopping power $S(E)$ was set to zero to recover the monoenergetic problem from Eq. (9). As a part of the asymptotic scalings, the Fokker–Planck scattering operator in Eq. (9) was written as the sum of two operators, and one of these was scaled as small compared to the other. This, coupled with other scalings, gave Eq. (14) as the lowest-order result in an asymptotic expansion.

Proceeding to one higher order, Larsen found the result

$$\rho(x, y, z) = (1 + \sigma_1 z)\, \tilde{\rho}(x, y, z) \tag{17}$$

where $\tilde{\rho}$ satisfies the Fermi equation, i.e.,

$$\frac{\partial \tilde{\rho}}{\partial z} = \frac{\sigma_1 z^2}{2} \nabla^2 \tilde{\rho} \tag{18}$$

Larsen then carried his analysis to still one higher order, and obtained

$$\rho(x, y, z) = [1 + \sigma_1 z + 2(\sigma_1 z)^2]\, \tilde{\rho}(x, y, z) \tag{19}$$

where in this order $\tilde{\rho}$ satisfies the equation

$$\frac{\partial \tilde{\rho}}{\partial z} = \left( \frac{\sigma_1 z^2}{2} + \frac{5\sigma_1^2 z^3}{3} \right) \nabla^2 \tilde{\rho} \tag{20}$$

We note that all of these results contain no explicit reference to the system thickness $T$. However, based upon the analysis, these results are meant to apply to a thin slab in the sense that $\sigma_1 T \ll 1$, since in the derivations of these results backscattering was neglected. The linear in $z$ term in Eq. (17) and the quadratic in $z$ term in Eq. (19) are, in fact, simply small-$z$ asymptotic expansions of the transverse integrated (over $x$ and $y$) solution.[7] For a particular case with $\sigma_1 T = 0.1$, Larsen showed by comparison with Monte Carlo benchmark results that the accuracy of his analysis improves as one retains more terms in the asymptotic expansion. Specifically, for the problem considered, he showed that Eqs. (19) and (20) are more accurate than Eqs. (17) and (18), which in turn are more accurate than the Fermi solution given by Eq. (14). Two other recent attempts at this problem had been reported[13, 14] just prior to the Larsen treatment, but both are algebraically complex and appear to give results inferior to those of Larsen in general three-dimensional geometry. However, these two approaches give quite good results in one-dimensional geometry.[15]

In this paper, we show that a hierarchy of descriptions for the integral quantity $\rho(\mathbf{r}, E)$ can be obtained relatively simply from the underlying transport equation or its Fokker–Planck representation, without the necessity of introducing any scalings in the scattering operator as in the Larsen treatment. In lowest order, we obtain a result more accurate than that of Fermi. In next order, our analysis predicts results slightly more accurate than the highest-order results reported by Larsen,[12] reproduced here as Eqs. (19) and (20). Further, our model development includes absorption and energy dependence in a straightforward way, and allows arbitrary spatial dependences for the interaction coefficients $\sigma$ and $\sigma_s$. Lastly, our procedure is quite straightforward and algebraically transparent.

A summary of the remainder of this paper is as follows. In Section 2 we apply our analysis to the Fokker–Planck description of the problem given by Eq. (9), and in Section 3 we give the corresponding analysis based upon the full transport description given by Eq. (6). In Section 4 we indicate that our equations for $\rho(\mathbf{r}, E)$ are easily solved, in the case of monoenergetic transport in a homogeneous medium, by a combination of Laplace and Fourier transform methods. This leads to explicit analytic solutions which we numerically compare in Section 5 with Monte Carlo benchmark results to assess the accuracy of our treatment.

## 2. THE FOKKER–PLANCK ANALYSIS

In this section we use the Fokker–Planck description given by Eq. (9) as the starting point of our analysis. This Fokker–Planck-based analysis is somewhat simpler than the corresponding analysis based upon the full transport description given by Eq. (6). The full transport equation analysis is given in the next section and, as we discuss there, the Fokker–Planck analysis gives valuable insight as to how to proceed in the corresponding treatment of Eq. (6).

We begin the Fokker–Planck treatment by rewriting Eq. (9) in the somewhat more compact form

$$\mu \frac{\partial \psi}{\partial z} + \varepsilon \omega \cdot \nabla \psi + \sigma_a \psi = \frac{\partial}{\partial E}(S\psi) + \frac{\sigma_1}{2}\left[ L\psi + \left(\frac{1}{1-\mu^2}\right)\frac{\partial^2 \psi}{\partial \phi^2}\right] \tag{21}$$

where the operator $L$ is defined by

$$L\psi = \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial \psi}{\partial \mu} \tag{22}$$

and the transverse component of $\boldsymbol{\Omega} \cdot \nabla \psi$ is given by

$$\omega \cdot \nabla \psi = (1-\mu^2)^{1/2}\left(\cos\phi \frac{\partial \psi}{\partial x} + \sin\phi \frac{\partial \psi}{\partial y}\right) \tag{23}$$

with the $\nabla$ operator here and henceforth meaning the transverse (in $x$ and $y$) gradient operator. We introduced a formal smallness parameter $\varepsilon$ in Eq. (21) to indicate the presumption that the transverse gradients are small. At the end of our considerations, $\varepsilon$ is to be set to unity. We assume analytic behavior of the solution $\psi$ in $\varepsilon$ according to

$$\psi = \sum_{n=0} \varepsilon^n \psi^{(n)} \tag{24}$$

Use of Eq. (24) in Eq. (21) gives, upon equating coefficients of $\varepsilon^n$,

$$\mu \frac{\partial \psi^{(n)}}{\partial z} + \sigma_a \psi^{(n)} = \frac{\partial}{\partial E}(S\psi^{(n)}) + \frac{\sigma_1}{2}\left[ L\psi^{(n)} + \left(\frac{1}{1-\mu^2}\right)\frac{\partial^2 \psi^{(n)}}{\partial \phi^2}\right] - \omega \cdot \nabla \psi^{(n-1)},$$

$$n = 0, 1, \ldots \tag{25}$$

with $\psi^{(-1)}=0$. The boundary conditions on Eq. (25) follow from Eqs. (2)–(4) as

$$\psi^{(0)}(z=0)=F\frac{\delta(1-\mu)}{2\pi}, \qquad \mu>0 \tag{26}$$

$$\psi^{(n)}(z=0)=0, \qquad \mu>0, \qquad n\geqslant 1 \tag{27}$$

$$\psi^{(n)}(z=T)=0, \qquad \mu<0, \qquad n\geqslant 0 \tag{28}$$

$$\psi^{(n)}(x=\pm\infty)=\psi^{(n)}(y=\pm\infty)=0, \qquad n\geqslant 0 \tag{29}$$

We consider this sequence of equations for $0\leqslant n\leqslant 2$.

We define $\hat{Q}(\mu)$ as the azimuthal integral of any quantity $Q(\mu,\phi)$, i.e.,

$$\hat{Q}(\mu)=\int_0^{2\pi} d\phi\, Q(\mu,\phi) \tag{30}$$

Integration of Eq. (25) with $n=0$, Eq. (26), and Eqs. (28) and (29) for $n=0$ over azimuth gives the equation

$$\mu\frac{\partial\hat{\psi}^{(0)}}{\partial z}+\sigma_a\hat{\psi}^{(0)}=\frac{\partial}{\partial E}(S\hat{\psi}^{(0)})+\frac{\sigma_1}{2}L\hat{\psi}^{(0)} \tag{31}$$

with boundary conditions

$$\hat{\psi}^{(0)}(z=0)=F\,\delta(1-\mu), \qquad \mu>0 \tag{32}$$

$$\hat{\psi}^{(0)}(z=T)=0, \qquad \mu<0 \tag{33}$$

$$\hat{\psi}^{(0)}(x=\pm\infty)=\hat{\psi}^{(0)}(y=\pm\infty)=0 \tag{34}$$

By inspection of the equations defining $\psi^{(0)}$, it is clear that $\psi^{(0)}$ is independent of the azimuthal angle $\phi$, and hence

$$\hat{\psi}^{(0)}=2\pi\psi^{(0)} \tag{35}$$

Since $\psi^{(0)}$ is independent of $\phi$, the azimuthal integral of Eq. (25) for $n=1$ gives

$$\mu\frac{\partial\hat{\psi}^{(1)}}{\partial z}+\sigma_a\hat{\psi}^{(1)}=\frac{\partial}{\partial E}(S\hat{\psi}^{(1)})+\frac{\sigma_1}{2}L\hat{\psi}^{(1)} \tag{36}$$

and Eqs. (27)–(29) show that all boundary conditions on $\hat{\psi}^{(1)}$ are homogeneous. Thus we immediately deduce by inspection that

$$\hat{\psi}^{(1)}(\mathbf{r},E,\mu)=0 \tag{37}$$

Equation (37) does not, of course, imply that $\psi^{(1)}=0$. In fact, $\psi^{(1)}\neq 0$ and it is this nonvanishing $\psi^{(1)}$ which gives rise to the transverse diffusion of the particle beam. This is seen by integrating Eq. (25) for $n=2$ over azimuth. The result is

$$\mu\frac{\partial\hat{\psi}^{(2)}}{\partial z}+\sigma_a\hat{\psi}^{(2)}=\frac{\partial}{\partial E}(S\hat{\psi}^{(2)})+\frac{\sigma_1}{2}L\hat{\psi}^{(2)}-\int_0^{2\pi}d\phi\,\omega\cdot\nabla\psi^{(1)} \tag{38}$$

The boundary conditions on Eq. (38) follow from Eqs. (27)–(29), and are all homogeneous. Thus it is the nonvanishing of the integral term involving $\psi^{(1)}$ in Eq. (38) which leads to a nonzero $\hat{\psi}^{(2)}$.

If we now multiply Eq. (36) by $\varepsilon$, Eq. (38) by $\varepsilon^2$, and add the results to Eq. (31), we find

$$\begin{aligned}\left(\mu\frac{\partial}{\partial z}+\sigma_a\right)&(\hat{\psi}^{(0)}+\varepsilon\hat{\psi}^{(1)}+\varepsilon^2\hat{\psi}^{(2)})\\&=\frac{\partial}{\partial E}[S(\hat{\psi}^{(0)}+\varepsilon\hat{\psi}^{(1)}+\varepsilon^2\hat{\psi}^{(2)})]+\frac{\sigma_1}{2}L(\hat{\psi}^{(0)}+\varepsilon\hat{\psi}^{(1)}+\varepsilon^2\hat{\psi}^{(2)})\\&\quad-\varepsilon^2\int_0^{2\pi}d\phi\,\omega\cdot\nabla\psi^{(1)}\end{aligned} \tag{39}$$

The $\hat{\psi}^{(1)}$ terms in Eq. (39) could, of course, be deleted by virtue of Eq. (37). Now, from Eq. (24) we have

$$\hat{\psi}=\hat{\psi}^{(0)}+\varepsilon\hat{\psi}^{(1)}+\varepsilon^2\hat{\psi}^{(2)}+O(\varepsilon^3) \tag{40}$$

and we can thus rewrite Eq. (39) as

$$\mu\frac{\partial\hat{\psi}}{\partial z}+\sigma_a\hat{\psi}=\frac{\partial}{\partial E}(S\hat{\psi})+\frac{\sigma_1}{2}L\hat{\psi}-\varepsilon^2\int_0^{2\pi}d\phi\,\omega\cdot\nabla\psi^{(1)}+O(\varepsilon^3) \tag{41}$$

The boundary conditions on Eq. (41) are clearly given by

$$\hat{\psi}(z=0)=F\delta(1-\mu),\qquad \mu>0 \tag{42}$$

$$\hat{\psi}(z=T)=0,\qquad \mu<0 \tag{43}$$

$$\hat{\psi}(x=\pm\infty)=\hat{\psi}(y=\pm\infty)=0 \tag{44}$$

If we define $\chi_x$ and $\chi_y$ as

$$\chi_x=\psi^{(1)}\cos\phi,\qquad \chi_y=\psi^{(1)}\sin\phi \tag{45}$$

we can rewrite Eq. (41) as

$$\mu \frac{\partial \hat{\psi}}{\partial z} + \sigma_a \hat{\psi} = \frac{\partial}{\partial E}(S\hat{\psi}) + \frac{\sigma_1}{2} L\hat{\psi} - \varepsilon^2 (1-\mu^2)^{1/2} \left( \frac{\partial \hat{\chi}_x}{\partial x} + \frac{\partial \hat{\chi}_y}{\partial y} \right) + O(\varepsilon^3) \quad (46)$$

In accordance with Eq. (30), $\hat{\chi}_x$ and $\hat{\chi}_y$ are given in terms of $\chi_x$ and $\chi_y$ as the corresponding azimuthal integrals, i.e.,

$$\hat{\chi}_r = \int_0^{2\pi} d\phi \, \chi_r, \qquad r = x, y \quad (47)$$

To obtain an equation for $\hat{\chi}_x$, we multiply the defining equation for $\psi^{(1)}$, namely Eq. (25) for $n=1$, by $\cos\phi$ an integrate over azimuth to get

$$\mu \frac{\partial \hat{\chi}_x}{\partial z} + \sigma_a \hat{\chi}_x = \frac{\partial}{\partial E}(S\hat{\chi}_x) + \frac{\sigma_1}{2} \left( L\hat{\chi}_x - \frac{\hat{\chi}_x}{1-\mu^2} \right) - \frac{(1-\mu^2)^{1/2}}{2} \frac{\partial \hat{\psi}^{(0)}}{\partial x} \quad (48)$$

In deriving Eq. (48) we have integrated the $\partial^2 \psi^{(1)}/\partial \phi^2$ term twice by parts. Since

$$\hat{\psi} = \hat{\psi}^{(0)} + O(\varepsilon) \quad (49)$$

we can rewrite Eq. (48) as

$$\mu \frac{\partial \hat{\chi}_x}{\partial z} + \sigma_a \hat{\chi}_x = \frac{\partial}{\partial E}(S\hat{\chi}_x) + \frac{\sigma_1}{2} \left( L\hat{\chi}_x - \frac{\hat{\chi}_x}{1-\mu^2} \right) - \frac{(1-\mu^2)^{1/2}}{2} \frac{\partial \hat{\psi}}{\partial x} + O(\varepsilon) \quad (50)$$

Since the $\hat{\chi}_x$ term in Eq. (46) is multiplied by an $\varepsilon^2$, the $O(\varepsilon)$ error in the $\hat{\chi}_x$ equation given by Eq. (50) gives an error of $O(\varepsilon^3)$ in Eq. (46), and Eq. (46) already contains errors of this order. The boundary conditions on $\hat{\chi}_x$ at $z=0$, $z=T$, $x=\pm\infty$, and $y=\pm\infty$ are all homogeneous. In a similar way, we obtain an equation for $\hat{\chi}_y$ by multiplying Eq. (25) for $n=1$ by $\sin\phi$ and integrating over $0 \leqslant \phi \leqslant 2\pi$. The result is

$$\mu \frac{\partial \hat{\chi}_y}{\partial z} + \sigma_a \hat{\chi}_y = \frac{\partial}{\partial E}(S\hat{\chi}_y) + \frac{\sigma_1}{2} \left( L\hat{\chi}_y - \frac{\hat{\chi}_y}{1-\mu^2} \right) - \frac{(1-\mu^2)^{1/2}}{2} \frac{\partial \hat{\psi}}{\partial y} + O(\varepsilon) \quad (51)$$

It is convenient for our subsequent analysis to eliminate $\hat{\chi}_x$ and $\hat{\chi}_y$ in favor of $\hat{\eta}_x$ and $\hat{\eta}_y$, defined as

$$\hat{\eta}_r = (1-\mu^2)^{1/2} \hat{\chi}_r, \qquad r = x, y \quad (52)$$

Thus our pertinent results, given by Eqs. (46), (50), and (51), are written, setting $\varepsilon = 1$ at this point and dropping the error order indicators,

$$\mu \frac{\partial \hat{\psi}}{\partial z} + \sigma_a \hat{\psi} = \frac{\partial}{\partial E}(S\hat{\psi}) + \frac{\sigma_1}{2} L\hat{\psi} - \left(\frac{\partial \hat{\eta}_x}{\partial x} + \frac{\partial \hat{\eta}_y}{\partial y}\right) \tag{53}$$

$$\mu \frac{\partial \hat{\eta}_r}{\partial z} + \sigma_a \hat{\eta}_r = \frac{\partial}{\partial E}(S\hat{\eta}_r) + \frac{\sigma_1}{2}\left[(1-\mu^2)^{1/2} L\left(\frac{\hat{\eta}_r}{(1-\mu^2)^{1/2}}\right) - \frac{\hat{\eta}_r}{1-\mu^2}\right] - \frac{(1-\mu^2)}{2}\frac{\partial \hat{\psi}}{\partial r}, \qquad r = x, y \tag{54}$$

with the operator $L$ given by Eq. (22). The boundary conditions at $z = 0$ on Eqs. (53) and (54) are

$$\hat{\psi}(z=0) = F\,\delta(1-\mu), \qquad \mu > 0 \tag{55}$$

$$\hat{\eta}_r(z=0) = 0, \qquad \mu > 0, \qquad r = x, y \tag{56}$$

We emphasize at this point that Eqs. (53)–(56) apply to a completely general medium, in that $\sigma_a$, $\sigma_1$, and $S$ are allowed to be arbitrary functions of $x$, $y$, and $z$. In the special case that these three quantities are independent of $x$ and $y$ ($z$ dependences are allowed), the three coupled equations given by Eqs. (53) and (54) can be reduced to a set of only two coupled equations. If we define

$$\hat{\eta} = \frac{\partial \hat{\eta}_x}{\partial x} + \frac{\partial \hat{\eta}_y}{\partial y} \tag{57}$$

then clearly Eq. (53) is written

$$\mu \frac{\partial \hat{\psi}}{\partial z} + \sigma_a \hat{\psi} = \frac{\partial}{\partial E}(S\hat{\psi}) + \frac{\sigma_1}{2} L\hat{\psi} - \hat{\eta} \tag{58}$$

An equation for $\hat{\eta}$ follows by differentiating Eq. (54) with respect to $r$ and adding the two results corresponding to $r = x$ and $r = y$. One finds

$$\mu \frac{\partial \hat{\eta}}{\partial z} + \sigma_a \hat{\eta} = \frac{\partial}{\partial E}(S\hat{\eta}) + \frac{\sigma_1}{2}\left[(1-\mu^2)^{1/2} L\left(\frac{\hat{\eta}}{(1-\mu^2)^{1/2}}\right) - \frac{\hat{\eta}}{1-\mu^2}\right] - \frac{1-\mu^2}{2}\nabla^2\hat{\psi} \tag{59}$$

with $\nabla^2$ representing the transverse Laplacian given by Eq. (15).

We now return to the general heterogeneous-medium results given by Eqs. (53)–(56), and treat these by assuming that $\hat{\psi}$ and $\hat{\eta}_r$ are peaked about $\mu = 1$. We introduce the moments

$$\psi_n = \int_{-1}^{1} d\mu (1-\mu)^n \hat{\psi}(\mu), \qquad n = 0, 1, \ldots \tag{60}$$

$$\eta_{rn} = \int_{-1}^{1} d\mu (1-\mu)^n \hat{\eta}_r(\mu), \qquad n = 0, 1, \ldots \tag{61}$$

The presumption is that

$$\psi_{n+1} \ll \psi_n, \qquad \eta_{r,n+1} \ll \eta_{rn} \tag{62}$$

because of the peakedness assumption. From Eqs. (55) and (56), the boundary conditions at $z = 0$ on these moments are

$$\psi_0(z=0) = F, \qquad \psi_n(z=0) = 0, \qquad n \geqslant 1 \tag{63}$$

$$\eta_{rn}(z=0) = 0, \qquad n \geqslant 0 \tag{64}$$

We develop differential equations for the $\psi_n$ and $\eta_{rn}$ by taking $(1-\mu)^m$, $m \geqslant 0$, moments of Eqs. (53) and (54), and using Eq. (62) to truncate the infinite set of moment equations. Specifically, we define the $N$th-order approximation as carrying the first $N+1$ moments of $\hat{\eta}_r(\mu)$ and the first $N+2$ moments of $\hat{\psi}(\mu)$. That is, we retain $\eta_{rn}$ for $0 \leqslant n \leqslant N$ and $\psi_n$ for $0 \leqslant n \leqslant N+1$. The reason for carrying one more $\psi_n$ moment than $\eta_{rn}$ moments is the factor of $(1-\mu^2)$ multiplying $\partial\hat{\psi}/\partial r$ in Eq. (54). We specifically are interested in $\psi_0$, which was previously denoted by $\rho$ in the introduction to this paper. It is $\psi_0$ which is needed to compute absorption and energy deposition rates.

We first consider the lowest-order approximation, namely $N = 0$. This corresponds to retaining the moments $\psi_0$, $\psi_1$, and $\eta_{r0}$. Formally dividing Eq. (53) by $\mu$, integrating the result over $-1 \leqslant \mu \leqslant 1$, and integrating the $(L\hat{\psi})/\mu$ term twice by parts gives

$$\frac{\partial \psi_0}{\partial z} + \sigma_a \int_{-1}^{1} d\mu \frac{\hat{\psi}}{\mu} = \frac{\partial}{\partial E}\left( S \int_{-1}^{1} d\mu \frac{\hat{\psi}}{\mu} \right) + \int_{-1}^{1} d\mu \left[ \frac{\sigma_1 \hat{\psi}}{\mu^3} - \frac{1}{\mu}\left( \frac{\partial \hat{\eta}_x}{\partial x} + \frac{\partial \hat{\eta}_y}{\partial y} \right) \right] \tag{65}$$

These formal operations are suspect at $\mu = 0$, but presumably can be considered rigorous in some distribution sense. However, such rigor is not required here. Our assumption of sharp peakedness of $\hat{\psi}$ and $\hat{\eta}$ about $\mu = 1$ implies that these functions are negligibly small (essentially zero) sufficiently far from $\mu = 1$, and thus the behavior of $1/\mu$ at $\mu = 0$ is irrelevant.

This peakedness assumption is implemented by expanding all algebraic functions of $\mu$ about $\mu = 1$. (These same comments apply throughout this paper whenever a division by $\mu$ is performed.) As we just indicated, we now expand $1/\mu$ and $1/\mu^3$ about $\mu = 1$, carrying first-order terms in $(1-\mu)$. This gives

$$\frac{\partial \psi_0}{\partial z} + \sigma_a(\psi_0 + \psi_1) = \frac{\partial}{\partial E}[S(\psi_0 + \psi_1)] + \sigma_1(\psi_0 + 3\psi_1) - \left(\frac{\partial \eta_{x0}}{\partial x} + \frac{\partial \eta_{y0}}{\partial y}\right) \quad (66)$$

We obtain the equation for $\partial \psi_1/\partial z$ in an analogous manner. We multiply Eq. (53) by $(1-\mu)/\mu$, integrate over $-1 \leqslant \mu \leqslant 1$, and algebraically manipulate to get

$$\frac{\partial \psi_1}{\partial z} + \sigma_a \psi_1 = \frac{\partial}{\partial E}(S\psi_1) + \sigma_1(\psi_0 + 3\psi_1) \quad (67)$$

The equation for $\partial \eta_{r0}/\partial z$ follows from Eq. (54) by dividing by $\mu$ and integrating over $-1 \leqslant \mu \leqslant 1$. After some algebraic manipulation, we find

$$\frac{\partial \eta_{r0}}{\partial z} + \sigma_a \eta_{r0} = \frac{\partial}{\partial E}(S\eta_{r0}) + \sigma_1 \eta_{r0} - \frac{\partial \psi_1}{\partial r}, \qquad r = x, y \quad (68)$$

Equations (66)–(68) constitute our lowest-order ($N = 0$) approximation for a general heterogeneous medium.

The next-order result ($N = 1$) follows by retaining $\psi_0$, $\psi_1$, $\psi_2$, $\eta_{r0}$, and $\eta_{r1}$ terms. Omitting the straightforward algebraic details, we find the seven equations

$$\frac{\partial \psi_0}{\partial z} + \sigma_a(\psi_0 + \psi_1 + \psi_2) = \frac{\partial}{\partial E}[S(\psi_0 + \psi_1 + \psi_2)] + \sigma_1(\psi_0 + 3\psi_1 + 6\psi_2) - \left(\frac{\partial \eta_{x0}}{\partial x} + \frac{\partial \eta_{y0}}{\partial y}\right) - \left(\frac{\partial \eta_{x1}}{\partial x} + \frac{\partial \eta_{y1}}{\partial y}\right) \quad (69)$$

$$\frac{\partial \psi_1}{\partial z} + \sigma_a(\psi_1 + \psi_2) = \frac{\partial}{\partial E}[S(\psi_1 + \psi_2)] + \sigma_1(\psi_0 + 3\psi_1 + 6\psi_2) - \left(\frac{\partial \eta_{x1}}{\partial x} + \frac{\partial \eta_{y1}}{\partial y}\right) \quad (70)$$

$$\frac{\partial \psi_2}{\partial z} + \sigma_a \psi_2 = \frac{\partial}{\partial E}(S\psi_2) + 2\sigma_1(2\psi_1 + 3\psi_2) \quad (71)$$

$$\frac{\partial \eta_{r0}}{\partial z}+\sigma_a(\eta_{r0}+\eta_{r1})=\frac{\partial}{\partial E}[S(\eta_{r0}+\eta_{r1})]+\sigma_1(\eta_{r0}+3\eta_{r1})$$
$$-\frac{1}{2}\left(2\frac{\partial \psi_1}{\partial r}+\frac{\partial \psi_2}{\partial r}\right), \qquad r=x, y \tag{72}$$

$$\frac{\partial \eta_{r1}}{\partial z}+\sigma_a\eta_{r1}=\frac{\partial}{\partial E}(S\eta_{r1})+\sigma_1(2\eta_{r0}+3\eta_{r1})-\frac{\partial \psi_2}{\partial r}, \qquad r=x, y \tag{73}$$

One could, of course, consider high-order approximations corresponding to $N \geqslant 2$. This is a straightforward algebraic procedure, following exactly along the lines we have just indicated for $N=0$ and $N=1$. In $N$th order, one obtains $3N+4$ coupled equations in the general case. We will not pursue these higher-order results in this paper. The final point we make in this section is that this set of $3N+4$ equations can be reduced to a set of $2N+3$ equations in the special case that $\sigma_a$, $\sigma_1$, and $S$ are independent of $x$ and $y$. In accordance with the discussion following Eq. (56), the two equations for $\eta_{xn}$ and $\eta_{yn}$ for a given index $n$ can be combined, by differentiating and adding, into a single equation for $\eta_n$, defined as

$$\eta_n=\frac{\partial \eta_{xn}}{\partial x}+\frac{\partial \eta_{yn}}{\partial y} \tag{74}$$

It is only the $\eta_n$ which occur in the $\psi_n$ equations. Thus in this special case, the four $N=0$ equations given by Eqs. (66)–(68) and the seven $N=1$ equations given by Eqs. (69)–(73) reduce to three and five equations, respectively. We will explicitly write these equations in Section 4 where we consider analytic and numerical results predicted by these models for a homogeneous medium.

## 3. THE FULL TRANSPORT EQUATION ANALYSIS

We now show that the same ideas we have applied to the Fokker–Planck equation in the last section can be applied to the full transport equation given by Eq. (6). The Fokker–Planck analysis just completed will be very useful as a guide to this more general analysis. We will point this out explicitly at the appropriate point in the development. Since the crux of our analysis is concerned with the treatment of the angular variables $\mu$ and $\phi$, we restrict our considerations in this section to the purely scattering monoenergetic problem. This keeps the equations relatively uncluttered, and the addition of absorption and energy dependence is entirely straightforward. The equation we start with is then, eliminating absorption and

energy dependence from Eq. (6), and decomposing the spatial gradient term as we did in Eq. (21),

$$\mu \frac{\partial \psi(\mathbf{r}, \mathbf{\Omega})}{\partial z} + \varepsilon \omega \cdot \nabla \psi(\mathbf{r}, \mathbf{\Omega}) + \sigma_{s0}(\mathbf{r})\, \psi(\mathbf{r}, \mathbf{\Omega}) = J\psi(\mathbf{r}, \mathbf{\Omega}) \tag{75}$$

Here $J$ is the integral scattering operator defined by

$$J\psi = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} \sigma_{sn} \int_0^{2\pi} d\phi' \int_{-1}^{1} d\mu'\, \psi(\mu', \phi') \times \left[ P_n(\mu)\, P_n(\mu') + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\mu)\, P_n^m(\mu') \cos m(\phi - \phi') \right] \tag{76}$$

As in Section 2, the $\nabla$ operator in Eq. (75) is the transverse (in $x$ and $y$) gradient operator, and $\varepsilon$ is a formal smallness parameter. In dealing with Eq. (75), we ignore the boundary conditions since their treatment is identical to that just given in connection with the Fokker–Planck equation.

Assuming an expansion of $\psi$ in $\varepsilon$ as before (see Eq. (24)], and following the same algebraic procedure, we arrive at the azimuthally integrated equations for $\hat{\psi}$, $\hat{\eta}_x$, and $\hat{\eta}_y$ given by

$$\mu \frac{\partial \hat{\psi}(\mu)}{\partial z} + \sigma_{s0} \hat{\psi}(\mu) = \sum_{k=0}^{\infty} \frac{2k+1}{2} \sigma_{sk} P_k(\mu) \int_{-1}^{1} d\mu'\, P_k(\mu')\, \hat{\psi}(\mu') - \left( \frac{\partial \hat{\eta}_x}{\partial x} + \frac{\partial \hat{\eta}_y}{\partial y} \right) \tag{77}$$

$$\mu \frac{\partial \hat{\eta}_r(\mu)}{\partial z} + \sigma_{s0} \hat{\eta}_r(\mu) = \sum_{k=1}^{\infty} \frac{2k+1}{2k(k+1)} \sigma_{sk} (1-\mu^2) \frac{dP_k(\mu)}{d\mu} \times \int_{-1}^{1} d\mu' \frac{dP_k(\mu')}{d\mu'} \hat{\eta}_r(\mu') - \frac{1-\mu^2}{2} \frac{\partial \hat{\psi}}{\partial r}, \qquad r = x, y \tag{78}$$

Equations (77) and (78) are the analogs of the Fokker–Planck results given by Eqs. (53) and (54) for $S = \sigma_a = 0$ (monoenergetic transport with no absorption). In writing Eq. (78) we have made use of

$$P_k^1(\mu) = (1-\mu^2)^{1/2} \frac{dP_k(\mu)}{d\mu} \tag{79}$$

Equation (78) can be put in a somewhat simpler form by using two identities for Legendre polynomials, namely

$$(1-\mu^2) \frac{dP_k(\mu)}{d\mu} = k[P_{k-1}(\mu) - \mu P_k(\mu)] \tag{80}$$

and

$$(2k+1)\,\mu P_k(\mu) = (k+1)\,P_{k+1}(\mu) + kP_{k-1}(\mu) \tag{81}$$

Use of these two identities in Eq. (78) yields

$$\mu\frac{\partial\hat{\eta}_r(\mu)}{\partial z} + \sigma_{s0}\hat{\eta}_r(\mu) = \sum_{k=1}^{\infty}\frac{k(k+1)}{2(2k+1)}\sigma_{sk}[P_{k-1}(\mu) - P_{k+1}(\mu)] \times \int_{-1}^{1} d\mu'\,\frac{P_{k-1}(\mu') - P_{k+1}(\mu')}{1-\mu'^2}\hat{\eta}_r(\mu') - \frac{1-\mu^2}{2}\frac{\partial\hat{\psi}}{\partial r}, \qquad r = x,\ y \tag{82}$$

It is also convenient in Eq. (77) to reconstruct $\hat{\psi}(\mu)$ from its Legendre moments according to

$$\hat{\psi}(\mu) = \sum_{k=0}^{\infty}\frac{2k+1}{2}P_k(\mu)\int_{-1}^{1} d\mu'\,P_k(\mu')\,\hat{\psi}(\mu') \tag{83}$$

Equation (83) allows us to rewrite Eq. (77) as

$$\mu\frac{\partial\hat{\psi}(\mu)}{\partial z} = \sum_{k=1}^{\infty}\frac{2k+1}{2}(\sigma_{sk}-\sigma_{s0})\,P_k(\mu)\int_{-1}^{1} d\mu'\,P_k(\mu')\,\hat{\psi}(\mu') - \left(\frac{\partial\hat{\eta}_x}{\partial x} + \frac{\partial\hat{\eta}_y}{\partial y}\right) \tag{84}$$

Equations (82) and (84) form the starting point for our subsequent analysis.

We treat these two equations just as we did Eqs. (53) and (54) in the Fokker–Planck analysis. That is, we assume that $\hat{\psi}(\mu)$ and $\hat{\eta}_r(\mu)$ are peaked about $\mu = 1$, introduce the $(1-\mu)^m$ moments as defined by Eqs. (60) and (61), and truncate the moment equations by neglecting $\psi_n$ for $n \geqslant N+2$ and $\eta_{rn}$ for $n \geqslant N+1$ in the $N$th-order approximation. We first consider the $N=0$ approximation. We expand $P_k(\mu)$ about $\mu = 1$ according to

$$P_k(\mu) = 1 - \frac{k(k+1)}{2}(1-\mu) + \cdots \tag{85}$$

and then Eq. (84) becomes

$$\mu\frac{\partial\hat{\psi}(\mu)}{\partial z} = \sum_{k=1}^{\infty}\frac{2k+1}{2}(\sigma_{sk}-\sigma_{s0})\,P_k(\mu)\left[\psi_0 - \frac{k(k+1)}{2}\psi_1\right] - \left(\frac{\partial\hat{\eta}_x}{\partial x} + \frac{\partial\hat{\eta}_y}{\partial y}\right) \tag{86}$$

We divide Eq. (86) by $\mu$ and integrate over $-1 \leqslant \mu \leqslant 1$, expanding $1/\mu$ about $\mu = 1$ and retaining quadratic terms, i.e.,

$$\frac{1}{\mu} = 1 + (1-\mu) + (1-\mu)^2 + \cdots$$

$$= \frac{10}{3} P_0(\mu) - 3P_1(\mu) + \frac{2}{3} P_2(\mu) + \cdots \tag{87}$$

where the Legendre polynomial terms are not to be interpreted as a truncated Legendre polynomial expansion of $1/\mu$. Rather, they are simply a regrouping of the first three terms in the power series expansion of $1/\mu$ about $\mu = 1$. At this point, one might enquire as to why three terms, not more and not less, were retained in this expansion of $1/\mu$. We shall return to this question at the end of our $N = 0$ development. The result of the algebraic procedure outlined just above Eq. (87) is

$$\frac{\partial \psi_0}{\partial z} = 3(\sigma_{s0} - \sigma_{s1})(\psi_0 - \psi_1) - \frac{2}{3}(\sigma_{s0} - \sigma_{s2})(\psi_0 - 3\psi_1) - \left(\frac{\partial \hat{\eta}_{x0}}{\partial x} + \frac{\partial \hat{\eta}_{y0}}{\partial y}\right) \tag{88}$$

We obtain the equation for $\partial \psi_1 / \partial z$ in an analogous manner. We multiply Eq. (86) by $(1-\mu)/\mu$, expand this term on the right-hand side about $\mu = 1$, carrying terms up to quadratic in $(1-\mu)$, and integrate over $-1 \leqslant \mu \leqslant 1$. The result is

$$\frac{\partial \psi_1}{\partial z} = 3(\sigma_{s0} - \sigma_{s1})(\psi_0 - \psi_1) - \frac{2}{3}(\sigma_{s0} - \sigma_{s2})(\psi_0 - 3\psi_1) \tag{89}$$

We obtain the equation for $\partial \eta_{r0} / \partial z$ from Eq. (82). We first note that Eq. (85) gives

$$P_{k-1}(\mu) - P_{k+1}(\mu) = (2k+1)(1-\mu) + \cdots \tag{90}$$

and thus Eq. (82) becomes, since we are neglecting $\eta_{rn}$ for $n \geqslant 1$,

$$\mu \frac{\partial \hat{\eta}_r(\mu)}{\partial z} + \sigma_{s0} \hat{\eta}_r(\mu) = \frac{\eta_{r0}}{4} \sum_{k=1}^{\infty} k(k+1)\, \sigma_{sk} [P_{k-1}(\mu) - P_{k+1}(\mu)]$$

$$- \frac{1-\mu^2}{2} \frac{\partial \hat{\psi}}{\partial r}, \qquad r = x, y \tag{91}$$

We divide Eq. (91) by $\mu$ and integrate over $-1 \leqslant \mu \leqslant 1$ to obtain, since we are neglecting $\psi_n$ for $n \geqslant 2$,

$$\frac{\partial \eta_{r0}}{\partial z} + \sigma_{s0}\eta_{r0} = \frac{\eta_{r0}}{4} \sum_{k=1}^{\infty} k(k+1)\,\sigma_{sk} \int_{-1}^{1} d\mu \frac{P_{k-1}(\mu) - P_{k+1}(\mu)}{\mu} - \frac{\partial \psi_1}{\partial r}, \qquad r = x, y \tag{92}$$

Finally, we expand $[P_{k-1}(\mu) - P_{k+1}(\mu)]/\mu$ about $\mu = 1$, again carrying terms quadratic in $(1-\mu)$. The final result for $\partial \eta_{r0}/\partial z$ is then

$$\frac{\partial \eta_{r0}}{\partial z} = (-\sigma_{s0} + 2\sigma_{s1} - \sigma_{s2})\,\eta_{r0} - \frac{\partial \psi_1}{\partial r}, \qquad r = x, y \tag{93}$$

Equations (88), (89), and (93) constitute our lowest-order ($N = 0$) approximation.

Let us consider these equations in the limit of very peaked scattering in the forward direction. In this limit, the Fokker–Planck formalism should be valid, and Eqs. (88), (89), and (93) should reduce to the $N = 0$ results of the last section. In general, $\sigma_{sk}$ is defined by

$$\sigma_{sk} = 2\pi \int_{-1}^{1} d\xi\, P_k(\xi)\,\sigma_s(\xi) \tag{94}$$

where $\sigma_s(\xi)$ is the monoenergetic scattering kernel, assumed to be peaked about $\xi = 1$. We approximate $P_k(\xi)$ in Eq. (94) by its expansion about $\xi = 1$ according to Eq. (85). This is the basis of the Fokker–Planck approximation.[1, 2, 5] We then find

$$\sigma_{sk} = \sigma_{s0} - \frac{k(k+1)}{2}\,\sigma_1 \tag{95}$$

with $\sigma_1$ defined by

$$\sigma_1 = 2\pi \int_{-1}^{1} d\xi (1-\xi)\,\sigma_s(\xi) \tag{96}$$

A comparison of Eq. (96) with Eq. (11) shows that $\sigma_1$ is just the Fokker–Planck angular deflection parameter in the monoenergetic case, since in this case

$$\sigma_s(E' \to E, \xi) = \sigma_s(\xi)\,\delta(E' - E) \tag{97}$$

Making use of Eq. (95) to eliminate $\sigma_{s1}$ and $\sigma_{s2}$ in Eqs. (88), (89), and (93) in favor of $\sigma_1$ gives

$$\frac{\partial \psi_0}{\partial z} = \sigma_1(\psi_0 + 3\psi_1) - \left(\frac{\partial \eta_{x0}}{\partial x} + \frac{\partial \eta_{y0}}{\partial y}\right) \tag{98}$$

$$\frac{\partial \psi_1}{\partial z} = \sigma_1(\psi_0 + 3\psi_1) \tag{99}$$

$$\frac{\partial \eta_{r0}}{\partial z} = \sigma_1 \eta_{r0} - \frac{\partial \psi_1}{\partial r}, \qquad r = x, y \tag{100}$$

These are, in fact, the $N=0$ Fokker–Planck equations for the purely scattering, monoenergetic problem [see Eqs. (66)–(68) with $\sigma_a = S = 0$].

Before leaving this $N=0$ development, we return to the question as to why we carried quadratic terms in $(1-\mu)$ in the expansion of $1/\mu$ [see Eq. (87)] and in similar expansions of other $\mu$-dependent functions. If lower order than quadratic terms are carried in these expansions, the resulting equations for $\psi_0$, $\psi_1$, and $\eta_{r0}$ do not reduce to the Fokker–Planck equations in the Fokker–Planck limit. Carrying terms above quadratic in $(1-\mu)$ leads to unnecessary algebraic complication, since these terms are not needed to reproduce the Fokker–Planck limit. Thus an independent Fokker–Planck development, as given in the last section, was needed as a necessary insight into the general transport equation development.

The next-order result ($N=1$) follows by retaining $\psi_0$, $\psi_1$, $\psi_2$, $\eta_{r0}$, and $\eta_{r1}$ terms. In this case, in treating the integration over $\mu'$ in Eq. (84), one needs to expand $P_k(\mu')$ about $\mu' = 1$ correct to quadratic terms in $(1-\mu')$ since the quadratic term gives rise to a $\psi_2$ contribution. That is, we use

$$P_k(\mu') = 1 - \frac{k(k+1)}{2}(1-\mu') + \frac{(k-1)(k)(k+1)(k+2)}{16}(1-\mu')^2 + \cdots \tag{101}$$

Futher, expansions of $\mu$-dependent terms about $\mu = 1$ prior to integration over $-1 \leqslant \mu \leqslant 1$ are carried in the $N=1$ case to terms cubic in $(1-\mu)$. This is necessary to give the correct Fokker–Planck limit. For example, the expansion of $1/\mu$ about $\mu = 1$ used is

$$\begin{aligned}\frac{1}{\mu} &= 1 + (1-\mu) + (1-\mu)^2 + (1-\mu)^3 + \cdots \\ &= \frac{16}{3}P_0(\mu) - \frac{33}{5}P_1(\mu) + \frac{8}{3}P_2(\mu) - \frac{2}{5}P_3(\mu) + \cdots\end{aligned} \tag{102}$$

We omit the straightforward algebraic details and simply give the results of the analysis. In terms of $a$ and $b_r$, defind as

$$a = \tfrac{33}{5}(\sigma_{s0}-\sigma_{s1})(\psi_0-\psi_1) - \tfrac{4}{3}(\sigma_{s0}-\sigma_{s2})(2\psi_0-6\psi_1+3\psi_2) + \tfrac{1}{5}(\sigma_{s0}-\sigma_{s3})(2\psi_0-12\psi_1+15\psi_2) \tag{103}$$

$$b_r = \tfrac{1}{5}(16\sigma_{s1}-15\sigma_{s2}+4\sigma_{s3})\,\eta_{r0} + (3\sigma_{s2}-2\sigma_{s3})\,\eta_{r1} \tag{104}$$

the seven $N=1$ equations are

$$\frac{\partial\psi_0}{\partial z} = a - \left(\frac{\partial\eta_{x0}}{\partial x}+\frac{\partial\eta_{y0}}{\partial y}\right) - \left(\frac{\partial\eta_{x1}}{\partial x}+\frac{\partial\eta_{y1}}{\partial y}\right) \tag{105}$$

$$\frac{\partial\psi_1}{\partial z} = a - \left(\frac{\partial\eta_{x1}}{\partial x}+\frac{\partial\eta_{y1}}{\partial y}\right) \tag{106}$$

$$\frac{\partial\psi_2}{\partial z} = a - (\sigma_{s0}-\sigma_{s1})(\psi_0-\psi_1) \tag{107}$$

$$\frac{\partial\eta_{r0}}{\partial z} = b_r - \sigma_{s0}(\eta_{r0}+\eta_{r1}) - \frac{1}{2}\left(2\frac{\partial\psi_1}{\partial r}+\frac{\partial\psi_2}{\partial r}\right), \qquad r=x,\ y \tag{108}$$

$$\frac{\partial\eta_{r1}}{\partial z} = b_r - (\sigma_{s1}\eta_{r0}+\sigma_{s0}\eta_{r1}) - \frac{\partial\psi_2}{\partial r}, \qquad r=x,\ y \tag{109}$$

In the Fokker–Planck limit for which Eq. (95) is valid, we have

$$a = \sigma_1(\psi_0+3\psi_1+6\psi_2) \tag{110}$$

$$b_r = \sigma_{s0}(\eta_{r0}+\eta_{r1}) + \sigma_1(\eta_{r0}+3\eta_{r1}) \tag{111}$$

and then Eqs. (105)–(109) reduce to

$$\frac{\partial\psi_0}{\partial z} = \sigma_1(\psi_0+3\psi_1+6\psi_2) - \left(\frac{\partial\eta_{x0}}{\partial x}+\frac{\partial\eta_{y0}}{\partial y}\right) - \left(\frac{\partial\eta_{x1}}{\partial x}+\frac{\partial\eta_{y1}}{\partial y}\right) \tag{112}$$

$$\frac{\partial\psi_1}{\partial z} = \sigma_1(\psi_0+3\psi_1+6\psi_2) - \left(\frac{\partial\eta_{x1}}{\partial x}+\frac{\partial\eta_{y1}}{\partial y}\right) \tag{113}$$

$$\frac{\partial\psi_2}{\partial z} = 2\sigma_1(2\psi_1+3\psi_2) \tag{114}$$

$$\frac{\partial\eta_{r0}}{\partial z} = \sigma_1(\eta_{r0}+3\eta_{r1}) - \frac{1}{2}\left(2\frac{\partial\psi_1}{\partial r}+\frac{\partial\psi_2}{\partial r}\right), \qquad r=x,\ y \tag{115}$$

$$\frac{\partial\eta_{r1}}{\partial z} = \sigma_1(2\eta_{r0}+3\eta_{r1}) - \frac{\partial\psi_2}{\partial r}, \qquad r=x,\ y \tag{116}$$

These seven equations are the correct $N=1$ Fokker–Planck equations for the purely scattering, monoenergetic problem [see Eqs. (69)–(73) with $\sigma_a = S = 0$].

Higher-order approximations, with index $N \geqslant 2$, are of course possible. As in the Fokker–Planck analysis, one obtains $3N+4$ equations in $N$th order, involving the scattering kernel expansion coefficients up to $\sigma_{s,N+2}$. In this $N$th-order analysis, one needs to carry the expansion of $P_k(\mu')$ about $\mu'=1$ in Eq. (84) to include terms of order $(1-\mu')^{N+1}$. In the expansion of $\mu$-dependent terms about $\mu=1$ prior to integration over $-1 \leqslant \mu \leqslant 1$, one carries terms of order $(1-\mu)^{N+2}$. We will not pursue these higher-order treatments in this paper. Finally, we again note that in $N$th order the $3N+4$ equations can be reduced to $2N+3$ equations if the $\sigma_{sk}$ for $0 \leqslant k \leqslant N+2$ are independent of $x$ and $y$. This reduction is accomplished just as in the Fokker–Planck case, per the discussion at the end of Section 2.

## 4. ANALYTIC RESULTS

In this section, we show that the equations we have developed can be solved analytically in the special case of monoenergetic transport in a homogeneous medium. Specifically, we consider the Fokker–Planck equations for both $N=0$ and $N=1$ in the no-absorption ($\sigma_a=0$) limit with no energy transfer ($S=0$) and for a homogeneous medium ($\sigma_1$ constant, independent of position). Since $\sigma_1$ is constant, we can use the reduced set of equations involving $\eta_n$ rather than $\eta_{xn}$ and $\eta_{yn}$ separately [see Eq. (74) and the surrounding discussion]. Thus the $N=0$ and $N=1$ models consist of three and five coupled equations, respectively. By a proper choice of the unit of distance, we can set $\sigma_1 = 1$ in these equations.

The $N=0$ Fokker–Planck equations, which result from a consolidation of Eqs. (66)–(68) from four equations to three, together with the appropriate boundary conditions [see Eqs. (63) and (64)], are given by

$$\frac{\partial \psi_0}{\partial z} = \psi_0 + 3\psi_1 - \eta_0 \tag{117}$$

$$\frac{\partial \psi_1}{\partial z} = \psi_0 + 3\psi_1 \tag{118}$$

$$\frac{\partial \eta_0}{\partial z} = \eta_0 - \nabla^2 \psi_1 \tag{119}$$

$$\psi_0(x, y, 0) = F(x, y) \tag{120}$$

$$\psi_1(x, y, 0) = \eta_0(x, y, 0) = 0 \tag{121}$$

where $F(x, y)$ is the prescribed boundary data. For any function $g(z)$, we introduce the Laplace transform $\bar{g}(s)$ according to

$$\bar{g}(s) = \int_0^\infty dz\, e^{-sz} g(z) \tag{122}$$

Laplace transforming Eqs. (117)–(119) gives

$$s\bar{\psi}_0 = \bar{\psi}_0 + 3\bar{\psi}_1 - \bar{\eta}_0 + F \tag{123}$$

$$s\bar{\psi}_1 = \bar{\psi}_0 + 3\bar{\psi}_1 \tag{124}$$

$$s\bar{\eta}_0 = \bar{\eta}_0 - \nabla^2\bar{\psi}_1 \tag{125}$$

Algebraically eliminating $\bar{\psi}_1$ and $\bar{\eta}_0$ from these three equations gives

$$\left(s - \frac{s}{s-3}\right)\bar{\psi}_0 = \frac{1}{(s-1)(s-3)}\nabla^2\bar{\psi}_0 + F \tag{126}$$

Before proceeding further with the $N=0$ analysis, we consider the corresponding $N=1$ equations in Laplace transform space. The $N=1$ Fokker–Planck equations and boundary conditions in physical space for a homogeneous medium with $\sigma_a = S = 0$ and $\sigma_1 = 1$ are given by [see Eqs. (63), (64), and (69)–(73)]

$$\frac{\partial\psi_0}{\partial z} = \psi_0 + 3\psi_1 + 6\psi_2 - \eta_0 - \eta_1 \tag{127}$$

$$\frac{\partial\psi_1}{\partial z} = \psi_0 + 3\psi_1 + 6\psi_2 - \eta_1 \tag{128}$$

$$\frac{\partial\psi_2}{\partial z} = 2(2\psi_1 + 3\psi_2) \tag{129}$$

$$\frac{\partial\eta_0}{\partial z} = \eta_0 + 3\eta_1 - \nabla^2\psi_1 - \frac{1}{2}\nabla^2\psi_2 \tag{130}$$

$$\frac{\partial\eta_1}{\partial z} = 2\eta_0 + 3\eta_1 - \nabla^2\psi_2 \tag{131}$$

$$\psi_0(x, y, 0) = F(x, y) \tag{132}$$

$$\psi_1(x, y, 0) = \psi_2(x, y, 0) = 0 \tag{133}$$

$$\eta_0(x, y, 0) = \eta_1(x, y, 0) = 0 \tag{134}$$

The corresponding equations in Laplace transform space are

$$s\bar{\psi}_0 = \bar{\psi}_0 + 3\bar{\psi}_1 + 6\bar{\psi}_2 - \bar{\eta}_0 - \bar{\eta}_1 + F \tag{135}$$

$$s\bar{\psi}_1 = \bar{\psi}_0 + 3\bar{\psi}_1 + 6\bar{\psi}_2 - \bar{\eta}_1 \tag{136}$$

$$s\bar{\psi}_2 = 2(2\bar{\psi}_1 + 3\bar{\psi}_2) \tag{137}$$

$$s\bar{\eta}_0 = \bar{\eta}_0 + 3\bar{\eta}_1 - \nabla^2\bar{\psi}_1 - \tfrac{1}{2}\nabla^2\bar{\psi}_2 \tag{138}$$

$$s\bar{\eta}_1 = 2\bar{\eta}_0 + 3\bar{\eta}_1 - \nabla^2\bar{\psi}_2 \tag{139}$$

Algebraically eliminating $\bar{\psi}_1$, $\bar{\psi}_2$, $\bar{\eta}_0$, and $\bar{\eta}_1$ from these equations yields a single closed equation for $\bar{\psi}_0$ given by

$$\left[s - \frac{s(s-6)}{s^2-9s-6}\right]\bar{\psi}_0 = \left[\frac{(s^2-7s+24)(s^2-9s-6)+6s(s-2)(s-6)}{(s^2-4s-3)(s^2-9s-6)^2}\right]\nabla^2\bar{\psi}_0 + F \tag{140}$$

In obtaining Eq. (140) we have neglected terms of order $\nabla^{2n}\bar{\psi}_0$ for $n>1$, which is consistent with neglecting, as we did, $\varepsilon^3$ and higher-order terms in the asymptotic expansion given by Eq. (24).

We see that both the $N=0$ result, Eq. (126), and the $N=1$ result, Eq. (140), can be written

$$\frac{1}{A(s)}\bar{\psi}_0(x, y, s) = B(s)\nabla^2\bar{\psi}_0(x, y, s) + F(x, y) \tag{141}$$

where, in each case,

$$A(s) = \frac{1}{s} + O(s^{-2}), \qquad B(s) = \frac{1}{s^2} + O(s^{-3}) \tag{142}$$

For any function $h(x, y)$, we define $\tilde{h}(k_x, k_y)$ as the double Fourier transform, i.e.,

$$\tilde{h}(k_x, k_y) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{i(k_x x + k_y y)} h(x, y) \tag{143}$$

The double Fourier transform of Eq. (141) is then

$$\left[\frac{1}{A(s)} + k^2 B(s)\right]\tilde{\bar{\psi}}_0(k_x, k_y, s) = \tilde{F}(k_x, k_y) \tag{144}$$

where $k^2 = k_x^2 + k_y^2$. Thus we have

$$\tilde{\tilde{\psi}}_0 = \left(\frac{A}{1 + k^2 AB}\right) \tilde{F} \tag{145}$$

Neglecting terms of $O(k^4)$, which is equivalent to again neglecting fourth-order and higher spatial derivatives, we can rewrite Eq. (145) as

$$\tilde{\tilde{\psi}}_0 = (A - k^2 A^2 B)\, \tilde{F} \tag{146}$$

We now define $Q(z)$ and $P(z)$ as the Laplace inverses of $A(s)$ and $A^2(s)\, B(s)$, respectively. We have

$$Q(z) = \mathscr{L}^{-1}[A(s)] = 1 + O(z) \tag{147}$$

$$P(z) = \mathscr{L}^{-1}[A^2(s)\, B(s)] = \frac{z^3}{6} + O(z^4) \tag{148}$$

Where $\mathscr{L}^{-1}$ denotes the inverse Laplace operator. We note that $Q(z)$ is just the one-dimensional (no $x$ and $y$ dependences) solution corresponding to unit boundary data, $F = 1$. Laplace inverting Eq. (146) then gives

$$\tilde{\psi}_0 = (Q - k^2 P)\, \tilde{F} = Q \tilde{F} e^{-k^2 \tau} \tag{149}$$

where we have again neglected terms of $O(k^4)$, and defined

$$\tau(z) = \frac{P(z)}{Q(z)} = \frac{z^3}{6} + O(z^4) \tag{150}$$

Fourier inverting Eq. (149) gives

$$\psi_0(x, y, z) = Q(z)\, \rho(x, y, z) \tag{151}$$

where $\rho$ is the Fourier inverse of $\tilde{F} \exp(-k^2 \tau)$, i.e.,

$$\rho(x, y, z) = \mathscr{F}^{-1}[\tilde{F}(k_x, k_y)\, e^{-k^2 \tau(z)}] \tag{152}$$

with $\mathscr{F}^{-1}$ denoting the two-dimensional inverse Fourier transform operator.

We now show that $\rho(x, y, z)$ defined by Eq. (152) satisfies the diffusion equation

$$\frac{\partial \rho}{\partial z} = D \nabla^2 \rho \tag{153}$$

with boundary condition

$$\rho(x, y, 0) = F(x, y) \tag{154}$$

and a diffusion coefficient $D$ given by

$$D(z) = \frac{d\tau(z)}{dz} = \frac{z^2}{2} + O(z^3) \tag{155}$$

We first note that Eqs. (153) and (154) can be written as

$$\frac{\partial \Theta}{\partial \tau} = \nabla^2 \Theta \tag{156}$$

$$\Theta(x, y, 0) = F(x, y) \tag{157}$$

where $\Theta(x, y, \tau) = \rho(x, y, z)$. Fourier transforming Eq. (157) and Laplace and Fourier transforming Eq. (156) gives

$$\tilde{\bar{\Theta}} = \frac{\tilde{F}}{s + k^2} \tag{158}$$

Laplace inverting Eq. (158) yields

$$\tilde{\Theta} = \tilde{F} e^{-k^2 \tau} \tag{159}$$

and thus $\Theta(x, y, \tau)$ is given by the Fourier inverse of the right-hand side of Eq. (159), i.e.,

$$\Theta(x, y, \tau) = \rho(x, y, z) = \mathscr{F}^{-1}[\tilde{F}(k_x, k_y)\, e^{-k^2 \tau(z)}] \tag{160}$$

Equation (160) agrees with Eq. (152), thus demonstrating that the function $\rho(x, y, z)$ in the solution for $\psi_0(x, y, z)$ given by Eq. (151) indeed satisfies the diffusion equation and boundary condition given by Eqs. (153) and (154).

Equations (153) and (154) are easily solved in terms of a Green's function. We define this Green's function $G(x, y, z; x', y')$ by the diffusion equation

$$\frac{\partial G}{\partial z} = D \nabla^2 G \tag{161}$$

with boundary condition

$$G(x, y, 0; x', y') = \delta(x - x')\, \delta(y - y') \tag{162}$$

The solution of the diffusion equation with general boundary data $F(x, y)$ is then given by

$$\rho(x, y, z) = \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \, G(x, y, z; x', y') \, F(x', y') \tag{163}$$

This Green's function is easily shown to be given by

$$G(x, y, z; x', y') = \frac{1}{4\pi\tau(z)} \exp\left[ -\frac{(x-x')^2 + (y-y')^2}{4\tau(z)} \right] \tag{164}$$

where

$$\tau(z) = \int_0^z dz' \, D(z') \tag{165}$$

An algebraically simple case is that corresponding to radially symmetric boundary data given by a Gaussian. That is, if

$$F(x, y) = \frac{1}{\pi v} e^{-r^2/v} \tag{166}$$

where $r^2 = x^2 + y^2$ and $v$ is a constant related to the width of the Gaussian, then Eq. (163) yields

$$\rho(x, y, z) = \frac{1}{\pi(4\tau + v)} e^{-r^2/(4\tau + v)} \tag{167}$$

Equation (167) expresses the well-known fact that the convolution of two Gaussians is also a Gaussian, with the variance given by the sum of the two input variances. We will use Gaussian boundary data in our numerical results to be given shortly.

We now apply these general considerations to our $N=0$ and $N=1$ Fokker–Planck models. Considering first $N=0$, we have [see Eqs. (126) and (141)]

$$A(s) = \frac{(s-3)}{s(s-4)}, \qquad B(s) = \frac{1}{(s-1)(s-3)} \tag{168}$$

$A(s)$ is easily inverted to give

$$Q(z) = \mathscr{L}^{-1} A(s) = \tfrac{1}{4}(3 + e^{4z}) \tag{169}$$

This has the small-$z$ expansion

$$Q(z) = 1 + z + 2z^2 + \tfrac{8}{3}z^3 + O(z^4) \tag{170}$$

We recall that $Q(z)$ is just the one-dimensional solution with boundary data $F = 1$, and this problem has been considered in detail by Börgers and Larsen[7] and Prinja and Pomraning.[15] The first three terms on the right-hand side of Eq. (170) agree with these one-dimensional results, but our cubic term is incorrect. It should be $20z^3/3$.[7, 15] Thus our $N = 0$ model predicts the correct one-dimensional solution up to and including quadratic terms in the small-$z$ expansion. From Eq. (168) we have

$$A^2(s)\, B(s) = \frac{(s-3)}{s^2(s-1)(s-4)^2} \tag{171}$$

Rather than inverting this exactly, we content ourselves with a small-$z$ expansion. We have

$$A^2(s)\, B(s) = \frac{1}{s^4} + \frac{6}{s^5} + O(s^{-6}) \tag{172}$$

which inverts to

$$P(z) = \mathscr{L}^{-1}(A^2 B) = \frac{z^3}{6} + \frac{z^4}{4} + O(z^5) \tag{173}$$

Thus

$$\tau(z) = \frac{P(z)}{Q(z)} = \frac{z^3}{6} + \frac{z^4}{12} + O(z^5) \tag{174}$$

and

$$D(z) = \frac{d\tau(z)}{dz} = \frac{z^2}{2} + \frac{z^3}{3} + O(z^4) \tag{175}$$

At this point, we can easily make contact with Larsen's work.[12] If we approximate $Q(z)$ and $D(z)$ by the leading-order terms in the small-$z$ expansions [see Eqs. (170) and (175)], i.e., if we use

$$Q(z) = 1, \qquad D(z) = z^2/2 \tag{176}$$

we reproduce Larsen's lowest-order result, originally given by Fermi[8, 9] [see Eq. (14) with $\sigma_1 = 1$]. If we use $D(z)$ as given by Eq. (176), but retain

the linear term in the small-$z$ expansion of $Q(z)$ as given by Eq. (170), i.e., if we use

$$Q(z) = 1 + z, \qquad D(z) = z^2/2 \tag{177}$$

our $N = 0$ analysis gives Larsen's next-order result [see Eqs. (17) and (18) with $\sigma_1 = 1$]. Larsen's highest-order result, reproduced here as Eqs. (19) and (20), corresponds in our notation to (with $\sigma_1 = 1$)

$$Q(z) = 1 + z + 2z^2, \qquad D(z) = \frac{z^2}{2} + \frac{5}{3} z^3 \tag{178}$$

We see that our $N = 0$ treatment reproduces the quadratic Larsen terms in the one-dimensional solution $Q(z)$ [compare Eqs. (170) and (178)], but gives a different cubic term in the diffusion coefficient $D(z)$ [compare Eqs. (175) and (178)]. Larsen's cubic term is correct, and we will find this term in our $N = 1$ approximation, which we now consider.

For $N = 1$ we have [see Eqs. (140) and (141)]

$$A(s) = \frac{(s^2 - 9s - 6)}{s^2(s - 10)} \tag{179}$$

$$B(s) = \frac{(s^2 - 7s + 24)(s^2 - 9s - 6) + 6s(s - 2)(s - 6)}{(s^2 - 4s - 3)(s^2 - 9s - 6)^2} \tag{180}$$

$A(s)$ is easily inverted to give

$$Q(z) = \mathscr{L}^{-1} A(s) = \tfrac{1}{25}(24 + 15z + e^{10z}) \tag{181}$$

This has the small-$z$ expansion

$$Q(z) = 1 + z + 2z^2 + \tfrac{20}{3} z^3 + \tfrac{50}{3} z^4 + O(z^5) \tag{182}$$

This one-dimensional result for $N = 1$ correctly gives the cubic term $(20z^3/3)$ which the $N = 0$ result did not, but the $z^4$ coefficient is incorrect (it should be 95/3).[7, 15] As shown elsewhere in the one-dimensional context,[15] one would obtain the correct $z^4$ coefficient if our analysis is carried to next order ($N = 2$).

From Eqs. (179) and (180) we have

$$A^2(s)\, B(s) = \frac{(s^2 - 7s + 24)(s^2 - 9s - 6) + 6s(s - 2)(s - 6)}{s^4(s - 10)^2 (s^2 - 4s - 3)} \tag{183}$$

For large $s$, Eq. (183) yields

$$A^2(s)\,B(s) = \frac{1}{s^4} + \frac{14}{s^5} + O(s^{-6}) \tag{184}$$

and this inverts to

$$P(z) = \mathscr{L}^{-1}(A^2B) = \frac{z^3}{6} + \frac{7}{12}z^4 + O(z^5) \tag{185}$$

We then find

$$\tau(z) = \frac{P(z)}{Q(z)} = \frac{z^3}{6} + \frac{5}{12}z^4 + O(z^5) \tag{186}$$

and

$$D(z) = \frac{d\tau(z)}{dz} = \frac{z^2}{2} + \frac{5}{3}z^3 + O(z^4) \tag{187}$$

Thus we see that our $N=1$ approximation reproduces the highest-order result of Larsen[12] if terms up to quadratic in the small-$z$ expansion of $Q(z)$, and terms up to cubic in the small-$z$ expansion of $D(z)$, are retained in our expansions [compare Eqs. (182) and (187) with Eqs. (19) and (20) with $\sigma_1 = 1$].

The same solution techniques we used here for the Fokker–Planck equations can, with equal case, be applied to the full transport equation $N=0$ and $N=1$ models developed in Section 3, but we do not consider the details here. However, there is one final item we do consider in concluding this paper. This is comparison of numerical results predicted by our $N=0$ and $N=1$ models with benchmark Monte Carlo results. This is considered in the next section.

## 5. NUMERICAL RESULTS

The class of problems we consider numerically is the same class we have already considered analytically, namely monoenergetic transport in a purely scattering, homogeneous medium. We take the scattering kernel to be exponential in form and given by

$$\sigma_s(\xi) = \frac{\sigma e^{-(1-\xi)/\lambda}}{2\pi\lambda(1-e^{-2/\lambda})} \tag{188}$$

where $\xi = \Omega' \cdot \Omega$ is the cosine of the scattering angle. The numerical values of the parameters $\lambda$ and $\sigma$ in Eq. (188) which we shall use are $\lambda = 0.001$ and $\sigma = 100$. With this value of $\lambda$, the scattering is very peaked in the forward direction, and Fokker–Planck description of scattering is valid.[5] Using these data, we compute the Fokker–Planck parameter $\sigma_1$ from Eq. (96) as, neglecting terms of $O(e^{-2/\lambda})$,

$$\sigma_1 = \sigma\lambda = 0.1 \tag{189}$$

and since the problems under consideration are purely scattering, $\sigma_1$ is just the transport cross section.

We take the slab to have a thickness $T = 1$, and accordingly $\sigma_1 T = 0.1$ and $\sigma T = 100$. Thus this slab has a thickness of 100 as measured in particle mean free paths, and a thickness of 0.1 as measured in transport mean free paths. It is $\sigma_1 T$ which must be small for the beam to remain peaked about $\mu = 1$.[9] Finally, we assume that the input beam at $z = 0$ is Gaussian in form and given by, with $r^2 = x^2 + y^2$,

$$F(x, y) = \frac{1}{\pi v} e^{-r^2/v} \tag{190}$$

Here $v$ is a specified parameter which is just twice the variance of this normal distribution. For any value of $v$, $F(x, y)$ is normalized to a unit area, i.e.,

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, F(x, y) = \int_0^{2\pi} d\theta \int_0^{\infty} dr\, r \frac{1}{\pi v} e^{-r^2/v} = 1 \tag{191}$$

The numerical results we consider are for $\psi_0(r, T)$, the angularly integrated intensity at $z = T = 1$, the exiting surface of the slab, as a function of the radial coordinate $r$. We use two different values of the parameter $v$ in the input Gaussian given by Eq. (190), namely $v = 0.2$ and $v = 0$. We compare Monte Carlo benchmark results (kindly supplied to us by C. Börgers) with the predictions of the Fermi model and our $N = 0$ and $N = 1$ Fokker–Planck models. The Monte Carlo results involved $10^7$ histories, and the statistical errors in $\psi_0(r, T)$ are estimated to vary roughly monotonically from about 0.1% for small $r$ ($0 \leqslant r \leqslant 0.4$) to about 3% for large $r$ ($1.2 \leqslant r \leqslant 1.6$).

Figures 1–3 give results corresponding to $v = 0.2$ in the input Gaussian given by Eq. (190). Figure 1 gives Monte Carlo benchmark results for $\psi_0(r, T)$, as well as the predictions of the three models. The Monte Carlo

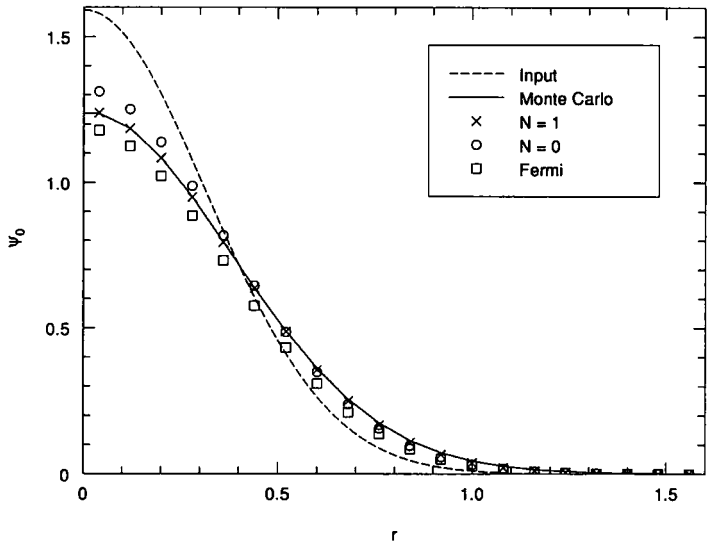


Fig. 1. The angularly integrated intensity $\psi_0$ for $v = 0.2$.

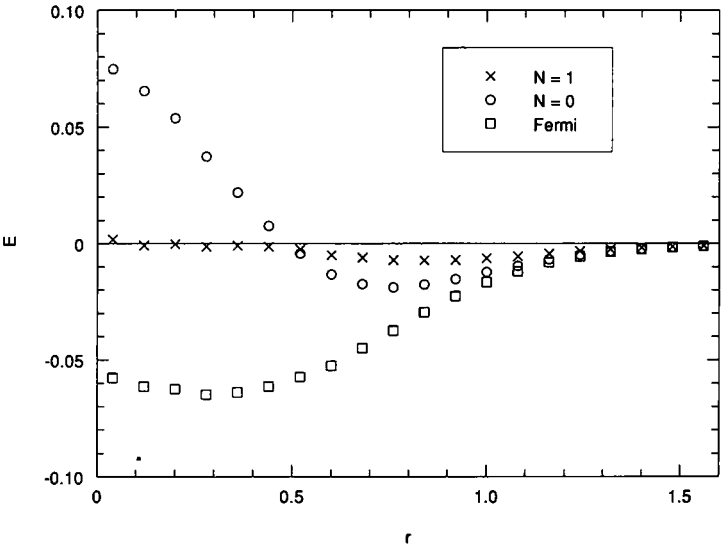


Fig. 2. The error $E$ for $v = 0.2$.

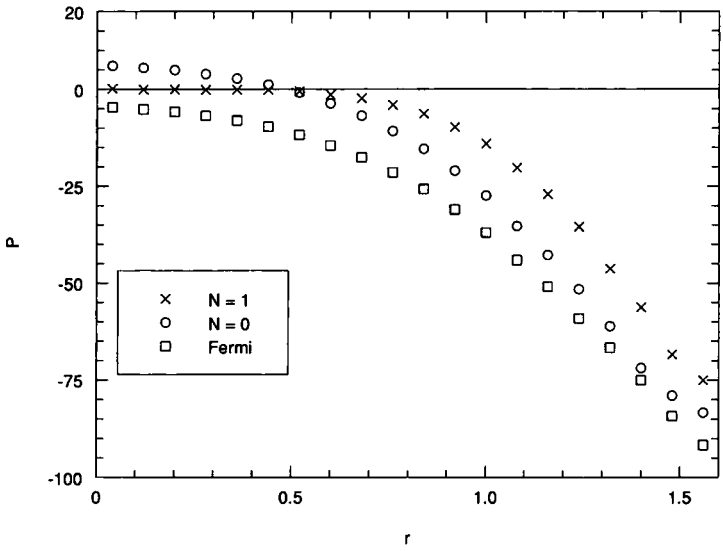


Fig. 3. The percentage error $P$ for $v = 0.2$.

results were computed with 20 radial tallying bins of $\Delta r = 0.08$ each. These 20 average (over the bin) values of $\psi_0(r, T)$ were plotted at the midpoint of each bin, and these 20 points were connected with straight lines. A horizontal line was drawn between $r = 0$ and $r = 0.04$ to indicate the zero slope at $r = 0$. The crosses, circles, and squares indicate the model results, again averaged over the Monte Carlo bins and plotted at the bin centers. We see that all three models perform quite well for this problem. For small $r$, for which $\psi_0(r, T)$ is substantial, the Fermi model is an underestimate of the true result, the $N = 0$ model overestimates the true result, and the $N = 1$ model, which appears to be exceedingly accurate, is roughly halfway between these two less accurate results. We have also indicated the input Gaussian, given by Eq. (190) with $v = 0.2$, as the dashed line in this figure. It is clear that the transverse spreading in this problem is quite small, and this accounts for the good performance of our $N = 0$ and $N = 1$ models. Our analysis was based upon the presumption of small transverse spreading.

To visually separate the predictions of the three models, Fig. 2 shows the error $E$ in each model, defined as

$$E = \psi_0(\text{model}) - \psi_0(\text{Monte Carlo}) \tag{192}$$

as a function of the radial coordinate $r$. Here we see the very small error, for each radial point, for the $N=1$ model. However, the percentage error $P$, defined as

$$P=\left[\frac{\psi_0(\text{model})-\psi_0(\text{Monte Carlo})}{\psi_0(\text{Monte Carlo})}\right]\times 100 \tag{193}$$

is substantial for large $r$ even in the $N=1$ model, as can be seen in Fig. 3. Large $r$ is associated with large angular deflections of the beam, and such deflections are treated poorly by all three models because of the presumption in their derivations of peakedness about $\mu=1$. Specifically, we see that all three models underestimate the true result for large $r$.

As was pointed out above, these $v=0.2$ results are very accurate because the transverse spreading of the beam is quite small in this case. A severe test of these models would be to set $v=0$, which corresponds to a true pencil beam of particles incident upon the slab. In this case, the transverse spreading of the beam will be large. Figures 4–6 give results for this case, and are analogous to Figs. 1–3, respectively. In this case, the Monte Carlo calculations used 20 tallying bins of equal width given by $\Delta r=\sqrt{0.1}/5=0.0632...$ All three figures indicate the robustness of the three models; all models are qualitatively correct even with large transverse beam spreading. It is worth noting that the Fermi model does not assume small transverse spreading,[9, 10] in contrast to our $N=0$ and $N=1$ models.

It can be seen from the figures that for small $r$, the Fermi solution is most accurate, whereas for large $r$, the $N=1$ model performs best. The error in the $N=1$ model changes sign twice as a function of $r$, whereas the Fermi model consistently underestimates the true result. The radially integrated error, which is just the error in the one-dimensional beam problem [corresponding to $F(x, y)=1$, rather than using Eq. (190)] is clearly the smallest for $N=1$ model, as discussed in the last section. This one-dimensional (radially integrated) error is about 12% for the Fermi model, and less than 1% for our $N=0$ and $N=1$ models. To summarize these $v=0$ results, the Fermi model, which underestimates the true solution for all $r$, is most accurate on a point-by-point basis for small and moderate values of $r$ where the magnitude of the intensity is substantial. In contrast, the $N=0$ and $N=1$ model results oscillate about the true solution, producing quite large errors for small and moderate $r$, but the average (integrated over $r$) error is very small. Thus, except for large $r$, the Fermi model gives a better radial spatial distribution of the particle collision rate, but the $N=0$ and $N=1$ models predict a much more accurate overall (integrated over $r$) collision rate.

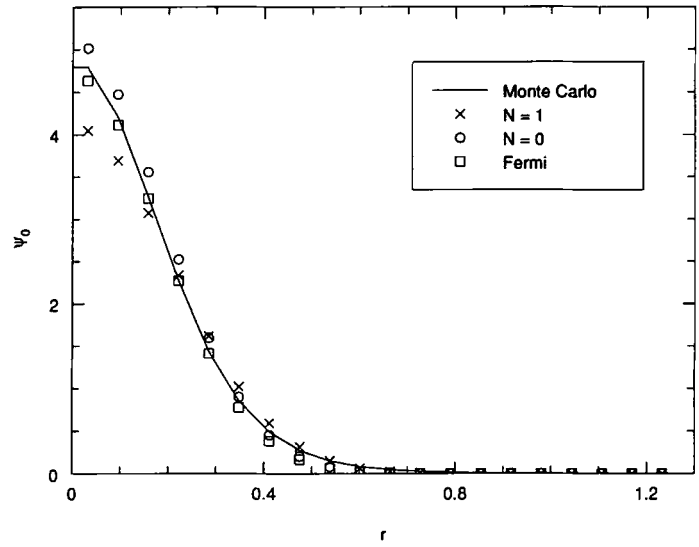


Fig. 4. The angularly integrated intensity $\psi_0$ for $v = 0$.

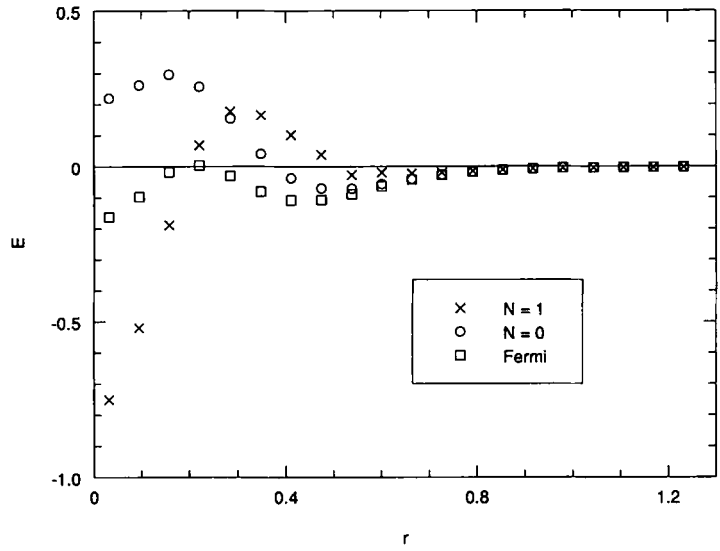


Fig. 5. The error $E$ for $v = 0$.

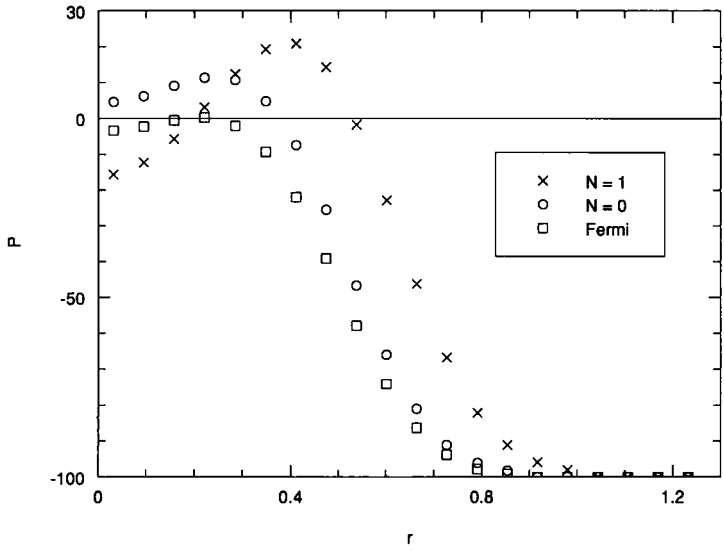


Fig. 6. The percentage error $P$ for $v = 0$.

The final item we take up in this section is to once again make contact with the earlier results of Larsen.[12] As discussed in the introduction to this paper, Larsen's lowest-order result is just Fermi's model. In next order, Larsen's result, given here by Eqs. (17) and (18), is comparable in numerical predictions to our $N = 0$ model, but somewhat less accurate (about 2% less accurate for the radially integrated result). Larsen's highest-order result, given here by Eqs. (19) and (20), is comparable to our $N = 1$ result, but again slightly less accurate (a fraction of a percent less accurate for the radially integrated result). The numerical accuracy values just given are based upon the two problems summarized in this section, but the relative accuracy statements apply qualitatively more generally.

In summary, the models of beam transport we have considered in this paper are strictly applicable to problems involving nearly collimated beams with small transverse spreading. For such problems, the models are very accurate. For problems involving large transverse spreading, our models are less accurate, but nonetheless are robust and lead to quite reasonable results. They are particularly accurate for the transverse integrated intensity. It should be emphasized that the analysis in this paper is very general in that it allows energy transfer and arbitrary spatial dependences in all cross sections in the underlying transport description. Further, the analysis was carried out for the general linear transport equation as well as its Fokker–Planck approximation, applicable to problems involving highly

forward peaked and small-energy-transfer scattering. The weakest ingredient in our analysis is the assumption of small transverse spreading of the beam, and any improved model will probably require relaxation of this assumption.

## ACKNOWLEDGMENTS

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